Spring 2016

A self-starting statistical control chart methodology for data exhibiting linear trend

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A SELF-STARTING STATISTICAL CONTROL CHART METHODOLOGY FOR DATA EXHIBITING LINEAR TREND

by

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A thesis submitted in partial fulfillment
of the requirements for the Master of Science
degree in Industrial Engineering in the
Graduate College of
The University of Iowa

May 2016

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CERTIFICATE OF APPROVAL

MASTER'S THESIS

This is to certify that the Master's thesis of

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ACKNOWLEDGEMENTS

I would like to thank my advisor, Yong Chen, for his continued guidance, knowledge, support and for his dedication and contributions to research in the field of Quality Engineering. I have learned more than I could ever imagine along the way and I value highly the knowledge and experience I have gained throughout my time at the University of Iowa. Lastly, to my family and friends thank you for your love and dedication to my best interests it has and will continue to mean the world to me.

ABSTRACT

Traditional quality control charts are designed to monitor and control a quality characteristic for processes with a stable in-control state in which enough data is available to estimate the process parameters prior to a production run. For many processes we desire the ability to monitor a quality characteristic that has an in-control state not stable such as a degradation or deterioration process that exhibits a linear trend as its in-control state. In addition, there are many times when sufficient sampling for in-control parameter estimation is not possible before the production run due to cost or collection time. We therefore desire a self-starting charting scheme that monitors both in-control and out of control linear trends. We present here the needed results so that a process with the in-control linear trend can be charted to detect slope and intercept shifts, when accurate information on in-control parameters is not available. We propose a *Q* chart scheme, a *EWMA Q* chart, and a *EWMA Q* chart with delay parameter *d* that utilizes results from statistical process control and linear regression. The developed control chart schemes are tested through simulation studies and applied to real data examples.

PUBLIC ABSTRACT

Quality goods and services are highly valued by today's consumer and companies that do not meet their quality expectations will suffer financially and lose customer loyalty. Therefore, it is in the best interest of the company to ensure that as many of their goods are of satisfactory quality as possible. The meaning of quality goods and services is well known but how do we ensure a process has the ability to consistently output a quality product. The answer lies in monitoring the process based on its statistical measures and quality control charts help us do just that.

Quality control charts help us define quality both mathematically and visually by charting a quantity of interest such as the mean value of a quality characteristic of the process. We estimate the mean and standard deviation of such a quality characteristic by collecting a sufficient amount of data from the process when it is said to be in-control. After gathering these estimates we have a baseline for which we can compare our future process observations against. We can then make a determination of the state of the process based on the number of standard deviations an observation lies from our in-control state. If we determine the process has deviated significantly from its in-control state, the engineer can take corrective action to resolve the quality issues.

Unfortunately, the time and effort taken to get enough observations up front to properly estimate the in-control parameters can prove to be difficult and costly. So the charting technique we propose has a self-starting property in which charting can begin at the start of a production run which helps to reduce the cost and time commitment taken to estimate the in-control process parameters. In addition, traditional control charts have another limitation in that they are limited to the case in which the quality characteristic of

interest has a stable mean. However, in many processes such as a degradation or deterioration process this mean is not stable and is instead approximated with a linear trend. Traditional control charts would signal immediately because they are designed for stable means, so we propose three methods that will effectively monitor deviations from a linear trend. Lastly, another problem we have encountered with traditional self-starting charts is that our in-control parameter estimates can be contaminated with out of control observations. Therefore, we introduce a delay parameter to our statistic that minimizes the in-control state contamination. In order to monitor the performance of such a process it is our desire to develop a chart that can detect in-control and out of control linear trend which we accomplished by applying linear regression results to a self-starting control chart statistic.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1 INTRODUCTION

In the manufacturing environment as well as many other engineering applications it is desirable to monitor the performance of a process so errors can be identified and corrected on the spot which will enable the engineer to minimize cost and nonconforming parts. Quality and Reliability literature documents a number of methods to accomplish this but we will focus on one specifically, statistical quality control charts. Control charts are often considered one of the more simplistic on-line statistical process control techniques, but can prove very powerful when used properly. The ultimate goal of any control chart is to reduce variability in the process and make overall process improvements. Control charts are a proven technique for improving productivity, preventing defects and unnecessary process adjustment, providing diagnostic information, and estimating process capability (Montgomery, 2005).

The basic construction of the control chart stems from a chosen quality characteristic of the process that is being analyzed. A process is said to be in statistical control if the process is operating with only chance causes of variation, which are an inherent part of the process (Montgomery, 2005). These chance causes of variation are random in nature and appear as white noise centered around the mean of the process of interest. When there is larger variation present, larger in magnitude than white noise, we suspect that there could be an assignable cause present that has caused this alarmingly large observation. The out of control state is reached by the occurrence of assignable causes which are mainly attributed to improper control or adjustments, operator errors, process faults, or defective raw materials. And it is the responsibility of the engineer to identify and remedy these problems (Montgomery, 2005). In the out of control state of the process a large proportion of the process output may not satisfy the quality requirements established by the engineer. Therefore, we desire a method to quickly detect the assignable causes of process shifts so corrective action can minimize the amount of non-conforming units manufactured, which is accomplished by the use of a control chart.

The basic statistical quality control chart is a graphical display of a quality statistic vs. the sample number or time. The chart is made up of a center line and upper and lower threshold values that if exceeded indicate the out of control state of the process being analyzed. The center line (CL) represents the average value of the quality characteristic of interest in the in-control state and is used as a reference value. The thresholds mentioned earlier are named the upper control limit and the lower control limit respectively. The upper control limit (UCL) and lower control limit (LCL) are chosen so that when the process is in-control nearly all the points will plot inside its bounds, typically UCL and the LCL are defined by the mean $\pm$ 3σ. Since the UCL and LCL are designed in this fashion a point that plots outside of the control limits is a cause for concern. Shown below in Figure 1 is an example of a control chart.

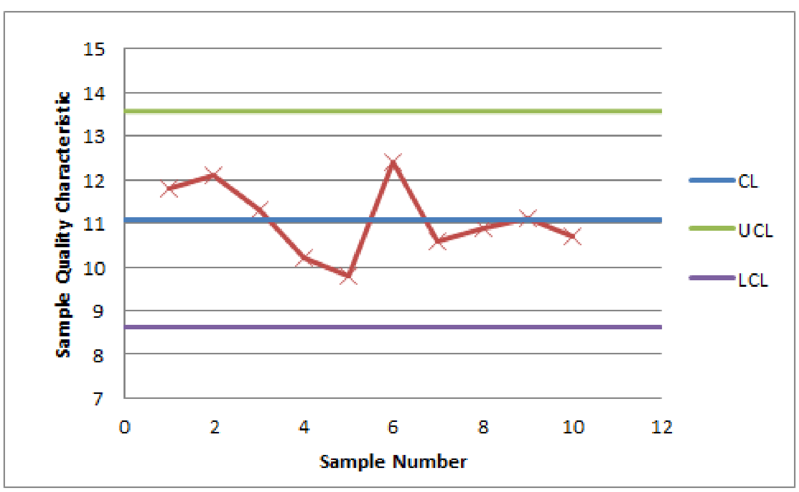


Figure 1: Example of a Statistical Quality Control Chart

The interpretation of the control chart follows from the statistical concept of hypothesis testing. Under the null hypothesis, which is our assumed state, the process is in-control. However, when a point plots outside of the control limits the null hypothesis is rejected, and we conclude the process is in the out of control state.

There are typically two main phases that characterize the statistical control charting process, Phase I and Phase II. Phase I is the preliminary set up of the control chart. In Phase I the in-control state must be reached and maintained to accurately assess the value of the in-control parameters such as the process mean. This is done by charting the process and systematically eliminating assignable causes repeatedly and then re-charting until a stable state has been reached. Once the stable state is reached the last part of Phase I is to estimate the statistics such as the mean and variance of the in-control process. Phase II involves comparing new observations to that of the in-control process distributions by designing appropriate center line, UCL, and LCL to quickly detect the change of state of the process.

To evaluate control chart performance in Phase II a few key concepts must first be understood. Similar to hypothesis testing, there are two types of errors in using a control chart. Concluding the process is out of control when it is actually in-control is defined as a Type I error. Likewise, concluding the process is in-control when it is actually out of control is a Type II error. Control charts are designed to minimize both of these errors but this minimization requires a balance of the two since there is a trade-off when one type of error prevention is favored over the other. For example, moving the control limits further apart decreases the Type I error but it necessarily increases the Type II error as a result. When assessing performance of any process it is extremely helpful to have a metric so the performance of one method can easily be compared to that of another. In Statistical Quality Control Charting this metric is called the Average Run Length, or ARL. The in-control ARL is defined as the average number of points that must be plotted before a point indicates an out of control condition, so the higher the in-control ARL the better the performance of the chart (Montgomery, 2005). The out of control ARL, ARL_1, is defined as the average amount of samples needed to detect a process shift. It is therefore desirable to have a short out of control ARL and a long in-control ARL so the engineer must find a chart that maximizes this ARL performance.

1.1 Literature Review

Now that the basics of control charts have been introduced we will outline the major categories of control charts relevant to this work in our literature review.

1.1.1 The Shewhart Control Chart

We start with an explanation of the simplest control chart, the Shewhart control chart which was developed by Walter A. Shewhart (Montgomery, 2005). The Shewhart control chart plots a statistic of interest such as sample mean against time or sample number. Shewhart charts use a center line, upper control limit, and lower control limit for process monitoring. The Shewhart control chart is used for single sample hypothesis testing in which the current state of the process is determined in-control or out of control based only on the status of the most recent observation. The quality statistic which is plotted on the chart as each observation occurs is connected in time sequence by the use of straight lines. In the case of Shewhart charts a quality statistic at time t is independent of all quality statistic observations greater than or less than t, which is a defining feature of Shewhart charts. Shewhart chart research paves the way for the rest of SPC control chart research to follow. The origins of SPC control charting started with the introduction of the Shewhart control chart by Walter A. Shewhart (Shewhart, 1931). Shewhart and Deming outline how to control and regulate variables and how to properly establish tolerance limits by taking a look at manufactured products (Shewhart, 1939). As more and more research has been conducted in this field numerous charting techniques have been developed all stemming from these original findings.

Shewhart control charts are useful in their simplicity and interpretability but are generally only effective in detecting large shifts in the process and are often much less effective when trying to detect smaller shifts (Montgomery, 2005). The Shewhart chart breaks down when small shift detection is desired because it analyzes the current status of the process solely off of the most recent point plotted. The Shewhart chart does not consider the points that were observed prior to the most recent observation. So at the time of each observation the state of the process is assessed by only analyzing one point, the most recent observation, which is its major downfall.

There are several charting methods that use the information from the previous observations of the process instead of considering only the most recent one, and we will outline two methods below.

1.1.2 The CUSUM Control Chart

The first of these is a cumulative sum control charting technique. The idea of the cumulative sum control chart, or *CUSUM* chart, is that it captures the history of the observations by charting the cumulative sum of the deviations of the mean of the sample observations from a target value μ_0 (Montgomery, 2005). The in-control state of the cumulative sum chart is then characterized as a random walk around zero. The cumulative sum chart is effective in small shift detection.

One of the first developments in *CUSUM* control charts was presented in Page (1954). Page (1954) applies a new methodology that monitors a quality characteristic's cumulative deviation from a target value. Ewan (1963) outlines the type of processes for which *CUSUM* charts are best applied and mentions several of the developments since their introduction. Lucas (1976) discusses v-mask *CUSUM* control schemes and showed they perform significantly better than Shewhart control charts in small shift detection through an example from the chemical industry. Hawkins (1981) extends the idea of *CUSUM* procedures for controlling the mean of the process to a procedure that will allow the engineer to chart the process variability and control the process variance. Lucas et. al (1982) proposed the Fast Initial Response (FIR) for *CUSUM* which facilitates a more rapid response to an initial out of control condition than that of a standard *CUSUM* procedure which helps in small shift detection. They recommend FIR be used at start-up or after an out of control signal occurs because it has been proven to be most effective at these times. ARL calculation methods for the *CUSUM* chart have been investigated by many researchers (Vance, 1986; Brook and Evans, 1972; Hawkins, 1992) and each offers their own approximation methods. Design considerations for the *CUSUM* chart are visited by (Gan, 1991). In this work Gan (1991) proposes a design strategy for *CUSUM* charts to easily determine the chart parameters of an optimal *CUSUM* chart.

1.1.3 The EWMA Control Chart

Another charting technique is the exponentially weighted moving average (*EWMA*) control chart. The *EWMA* chart assigns weights to the previously observed points in order to represent the process's history. The *EWMA* chart is set up so it assigns the largest weight to the most recent observation and assigns an exponentially decreasing weight to each subsequent older observation. The parameter that must be established by the engineer is λ which is commonly called the smoothing constant. The smoothing constant, λ, is established as the weight given to the most recent rational subgroup mean and the statistic decays in value from there with time. The weights are set up so they sum to unity. To see why the *EWMA* chart would be more effective in Phase II charting we compare the Shewhart chart methodology to the *EWMA* chart. The Shewhart chart assigns a weight of 1 to the most recent observation and assigns a weight of zero to all the prior observations, so it effectively does not consider any of the prior observations since they are assigned a weight of zero instead of an exponentially decaying weight with age.

Like the *CUSUM* chart the *EWMA* chart has been studied extensively in the Quality Engineering field. Roberts (1959) introduced the ideas of what has become the *EWMA* chart by weighting the observations in geometric progression from the most recent observation and then comparing this technique against traditional moving average performance. A procedure for numerical approximation of the ARL's of *EWMA* charts was devised by Robinson et al (1978) by expressing the generation of successive *EWMA*s as an autoregressive process. Crowder (1987) offered a more straight-forward less computationally intensive method for approximating ARL's associated with *EWMA*s by using an integral equation approach. Another advantage of *EWMA* charts is their robustness to non-normality. Borror, Montgomery and Runger (1999) investigated the *EWMA*'s robustness to non-normality through an ARL performance comparison to the Shewhart individuals chart. Crowder (1989) proposes design recommendations for

EWMA charts through plotting optimal smoothing parameters and control limit constants. Lucas and Saccucci (1990) propose a fast initial response feature for increased sensitivity for start-up problems and also propose a technique that combines Shewhart and *EWMA* schemes to detect both small and large shifts (Lucas & Saccuci, 1990).

1.1.4 The Problem of In-Control Parameter Estimation

A problem with all the control charting methods presented so far is the difficulties that arise from parameter estimation. Parameter estimation is a step in Phase I of control charting that involves gathering enough samples to estimate the quality characteristic of interest. Jensen et al. (2006) discusses the effects of parameter estimation extensively as it relates to control chart performance. They showed that when the number of reference samples is small, control charts with estimated parameters result in a large bias and consequently decrease the sensitivity of the chart when detecting process shifts. This presents a problem for Shewhart, *EWMA*, and *CUSUM* charts when short production runs are considered since using estimates in place of known parameters has been found to cause too many false alarms if sufficiently large in-control samples are not attained (Quesenberry, 1993; Jones et. al, 2001; Jones, 2004). Large calibration samples can be costly and problematic and in very short production runs the engineer cannot establish parameter estimates in time in order to chart the process in its initial stages. For an accurate parameter estimation the engineer requires a large number of in-control samples typically 25 or 30 calibration samples of size 4 or 5 each, which take a large amount of time to gather (Koning, 1997). Since many engineering environments are dynamic and fast-paced with short production runs and many changeovers, it may not be feasible to wait long enough to collect a sufficient number of samples. The engineer wants to minimize the amount of parts that are produced before charting has begun and monitor the process starting from its initial stages so they can begin identifying and removing assignable causes earlier rather than later.

The problem of investing significant time into parameter estimation in Phase I is resolved by what is called the self-starting control chart. Self-starting charts update the process parameter of interest recursively with every new observation. This allows the engineer to bypass the extensive sampling in the early stages of chart development which saves time and money and makes charting in short runs feasible. Self-starting methodologies can be applied to any of the charts outlined previously. For example, self-starting Cumulative sum charts and self-starting *EWMA* charts have been utilized quite often in recent Quality literature. Self-starting charts offer the advantage that they update the parameter estimates with the new observations and check the out of control conditions at the same time which is very desirable in developing any new charting methodologies.

1.1.5 The Self-Starting Control Chart

The self-starting control chart has many other advantages that improve the engineer's ability to control a process. One such advantage is the ability to chart in real time and essentially with the first units of production. Charting in real time at the beginning of a manufacturing run allows the engineer to correct assignable causes sooner and prevent non-conforming units from being manufactured. Another advantage related to parameter estimation is the ability to chart even when the exact in-control mean and standard deviation of the process are unknown. By using a self-starting chart the engineer can still determine if a shift has occurred from the conditions obtained at process startup without knowing the exact parameter values. The chart uses the past observations to estimate the in-control process parameters. The estimation errors of the process parameters based on the limited past observations are taken into account appropriately when it decides if a new observation is out of control.

The most basic self-starting methodology that we will focus on is the self-starting *Q* chart (Quesenberry, 1991). *Q* charts are Shewhart-type self-starting chart techniques. The *Q* chart estimates the in-control process mean and variance by iteratively calculating the sample mean and variance based on the $n-1$ observations seen before it. It can be shown that if the observations follow normal distribution, the statistic follows a *t*-distribution, which can be further transformed into a standard normal distribution. So the points can be plotted on a standardized normal control chart (Quesenberry, 1991). This transformed statistic is referred to as a *Q*-statistic and is obtained by utilizing the classical probability integral transformation Fisher (1930) and the conditional probability integral transformation O'Reilly (1973) to transform the original statistic into one that follows a standard normal distribution.

We will now take a closer look at the major findings in self-starting chart literature. Self-starting charts prove to be especially useful in the initial start-up phases as well as the short run production case when the exact value of the process parameters is not known. Hawkins (1987) initially proposed a self-starting *CUSUM* scheme that utilizes two pairs of *CUSUM*s one for monitoring the location of the process and the other for monitoring the spread. Quesenberry (1991) proposes the self-starting *Q* chart described earlier for both the mean and variance which applies transformations to the quality statistic so it can be plotted on standard normal control charts. Quesenberry

(1995) extends upon his original *Q* chart methodology and proposes the *CUSUM* of *Q* statistic. He et al (2008) proposes solutions for the bias of Shewhart *Q* charts that result when the out-of-control ARL is larger than the desired in-control ARL. *Q* statistics and the *Q* statistics applied to the residuals of a time series model are investigated in Kawamura et. al (2013) which focuses on short runs and auto-correlated data. Zantek (2006) offered an in depth analysis and improved design techniques for the *CUSUM* of *Q* statistics. Design techniques for self-starting charts were also visited in depth by Jones (2002). Since using parameter estimates with design procedures intended for known parameters can lead to worse charting performance Jones relaxes the assumption of known parameters and offers design procedures for the *EWMA* chart that improve the initial parameter-estimate charting performance.

Tsiamyrtzis and Hawkins (2008) use a Bayesian sequentially updated framework combined with a *EWMA* formulation to detect jumps in the start-up phase. The model process mean is represented as an autoregression with informative priors assumed from some prior information about the in-control process parameters. Using the idea of informative priors is logical since in most SPC applications there is some prior known information that provides the engineer some intuition about the process before charting has begun (Tsiamyrtzis and Hawkins, 2008). A self-starting control chart proposed by Li et al. (2010) also uses a *EWMA* procedure and combines it with likelihood ratio test to monitor the process mean and variance simultaneously when the process parameters are not known prior to start up. Capizzi and Masarotto (2012) develop what they call the *ACUSCORE* control chart which accounts for dynamic patterns in the process mean by utilizing an adaptive *EWMA* for a *CUSCORE* chart.

Thus far we have only looked at works that involve self-starting methodology in the univariate case. However, if we wish to chart two related quality statistics such as inner and outer diameters of a type of parts from the process startup, multivariate self-starting quality control charts are the best method. The multivariate case is discussed in several papers. Hawkins et al (2007) develops a multivariate equivalent of univariate self-starting charts, when it is desired to monitor two or more related quality characteristics for location and scale. Multivariate self-starting charts are also investigated by (Sullivan et al., 2002; Maboudou-Tchao et al., 2011; Capizzi et al., 2010).

1.1.6 Control Charts for Linear Trend

One noteworthy gap when considering the charting procedures we have outlined so far is that we have not considered the situation in which the process follows a linear trend as the in-control state. Representing the relationship between explanatory variables and a response variable with a linear equation is a common practice in the Engineering sciences. The engineer plots the explanatory variable against the response variable and models the relationship with a line of best fit that represents the expected value of the response variable at each level of the explanatory variable. The slope and intercept parameters are often estimated by linear regression which commonly uses the method of least squares for parameter estimation. Control charts for linear trends have been used to monitor such a process that assumes a specific linear trend in the in-control state, but none of these have incorporated self-starting methodologies. As the process exhibits a different trend or a point significantly deviates from the expected trend of the process, the process can be considered out of control.

Charts intended for a linear trend in-control state are very useful for monitoring processes whose output can be approximated by a linear model. Mansfield and Wein (1958) implemented a regression control chart that uses the residuals from a regression of cost on output that is able to detect days in which costs are unusually high or low. Mandel (1969) proposes a similar regression control chart to control a varying instead of a stable average of mail traffic in the postage industry. Ling et al. (1990) propose a cumulative student *t*-statistic based on regression residuals to control a photolithographic work cell process that utilizes a feed forward control methodology to update parameter estimates. Koksalan et al. (1999) propose a regression model using SPC methodologies for identifying missing independent variables in his beer demand study. Shu et al. (2004) investigates the run length performance of the regression control chart with estimated parameters. Zeng and Zhou (2007) use regression adjustment methods in monitoring multistage manufacturing processes by using the residuals from least squares linear regression. Sulek (2008) proposes a regression control chart based on least absolute value regression and finds his method is more sensitive than the traditional least squares regression technique to process shifts. There are several other charting applications that involve the use of linear models.

Mahmoud and Woodall (2004) considered the case when a quality characteristic can be modeled by a linear function and proposed a multiple regression model with use of indicator variables in Phase I analysis. Brown et al. (1975) studied how regression relationships change over time and proposed *CUSUM* and *CUSUM* of squares methodologies of recursive residuals to detect change in the original regression model. Bissell (1984) used *CUSUM*s to monitor linear trend in a target value and made ARL determinations using a non-homogeneous Markov Chain approach to handle a transition matrix that changes over time. Bissell warns about the accuracy of his ARL calculations and Gan (1992) proposes a numerical based method that generalizes the integral equation from Page (1954) and offers optimal design suggestions that improve the accuracy of the ARL estimation greatly. Wasserman et al. (1993) adapts the traditional *EWMA* formulation to include an additional term to model linear trends and utilizes a Bayesian estimation framework which allows for the inclusion of prior process knowledge.

Charting linear profiles is another related topic that has received recent attention and is discussed in Woodall et al. (2004) in great detail. Linear profile monitoring is used to monitor data with multiple observations cycle by cycle instead of by individual observation. Fahmy et al. (2006) proposes an MLE procedure for identifying the change point in processes subject to a linear trend that outperforms change point methods based on *EWMA* and *CUSUM* for estimating the actual drift time. Zou et al. (2006) utilizes a sequential change-point formulation which allows for simultaneous updating parameters and evaluation of out of control conditions. They then use the results proposed by Hawkins and Zamba (2005) to construct their chart with the maximal standardized likelihood ratio statistic, which they call the LRT chart that is able to detect shifts in the intercept, slope, and standard deviation of linear profiles. Phase I applications of linear profiles were analyzed in Stover and Brill (1998) Mestek et al. (1994) and Mahmoud et al. (2007). For phase II specific applications Kang and Albin (2000) propose two methods, the first involves a multivariate T^2 chart based on least squares estimators of the intercept and slope and the second uses statistics based on the successive samples of n deviations from the in-control line. Kim et al. (2003) proposed using the estimated regression coefficients of a linear function from each sample to construct two separate *EWMA* charts, one for the slope and the other for the intercept.

1.2 Motivation for our Methodology

The problem we wish to solve in this thesis is to design a self-starting control chart to monitor a process which follows a linear trend in its in-control state. To the best of our knowledge, the self-starting chart for a process that exhibits a linear trend in its in-control state has not been systematically studied in the literature. A self-starting chart such as this will be useful for short-run processes where the observations follow a linear trend or change with an explanatory variable in its in-control state. Another important application of such a self-starting control chart is to monitor degradation signals. Imagine a degradation or deterioration process where the performance of a machine or process degrades following a specific linear trend in the in-control state. In a typical degradation process the predicted linear equation only accurately represents the process in the early stages of its lifetime. After significant degradation has set in, the degradation rate may become more rapid and the process will follow a different often more rapid trend. Once the process has reached this more rapid degradation it is no longer considered in-control, and failure is typically imminent. It would therefore be extremely useful to detect this drift away from the original estimated linear trend so the engineer can be alerted that failure due to degradation could be occurring. For such degradation signals, historical data on the in-control state are not available. It is necessary to use a self-starting charting scheme to monitor the change of the degradation rate. The goal of this thesis is to develop a self-starting control chart technique that will monitor a process with unknown in-control parameters at startup and a linear trend as its in-control state that will be able to detect when the process shifts to the out of control state where the trend is significantly deviated from the in-control state. We present three control charting methods to solve this problem: a Shewhart chart based on an extension of the Q-statistics to the cases of linear trends, a EWMA chart, and an improved EWMA chart implementing a delay parameter.

In Chapter 2 we will provide the background knowledge necessary in order to understand the problem we wish to solve and our proposed methodologies. In Chapter 3 we will show our proposed methodologies for solving the linear trend detection problem, conduct a simulation study, and apply our charting methods to battery degradation data. Finally, in Chapter 4 we will discuss the findings from our results, offer a summary of the work we conducted, and lastly indicate opportunities for future work.

CHAPTER 2 BACKGROUND

A critical part of our solution lies within the ability to apply a self-starting methodology to our charting technique. In SPC literature there have been several different self-starting schemes proposed. One of the first and most proven of those is the self-starting Q chart (Quesenberry, 1991). Quesenberry's Q chart is a self-starting Shewhart-type control chart that has proven very useful because of its performance and simplicity. In this work we adapt the Q statistic in order to detect shifts in the in-control linear trend a process follows.

2.1 *Q* Chart Methodology

Quesenberry's Q chart allows for both the mean and variance of a process to be maintained from the start of production whether or not the value of the parameters is known prior to the run. The Q chart utilizes the information from the previous observations to iteratively estimate the mean and standard deviation of the process after each observation. The Q statistic follows the basic conventions of the standard Shewhart chart statistic by subtracting the mean from the current observation and dividing this difference by the standard deviation.

Due to estimation error when the in-control parameters are not known the statistic may not follow a standard normal distribution however, transformations such as the classic probability integral transformation can be applied which still preserves the information contained in the statistic but transforms it so it is normally distributed. We will now describe the methodology along with the necessary assumptions in greater detail.

2.1.1 Assumptions and Essential Equations of the Q Chart

Let $X_1, X_2, \ldots$ represent measurements from a process observed in time and assume these values are independently and identically distributed coming from a normal distribution with mean μ and variance σ^2. Quesenberry considers four cases in which some of these parameters are known and others are unknown prior to the run. When the in-control mean and variance are unknown prior to starting the run we must have a method for estimating them. The sample mean and sample variance equations allow us to estimate the in-control mean Eq. (2-1) and variance Eq. (2-2) based on the previous observations. The following equations are used in the development of the Q statistic and are our method for estimating the in-control mean and variance.

$$\bar{X}_r = \frac{1}{r}\sum_{j=1}^{r} X_j \tag{2-1}$$

$\bar{X}_r$ is the sample mean calculated after the rth observation and X_j is the jth observation in $\mathrm{j} = 1 \ldots \mathrm{r}$. Likewise the in-control variance is estimated by the sample variance equation.

$$S_r{}^2 = \frac{1}{r-1}\sum_{j=1}^{r}(X_j - \bar{X}_r)^2 \tag{2-2}$$

$S_r{}^2$ represents the sample variance calculated after the rth observation X_j is the jth observation in $\mathrm{j} = 1 \ldots \mathrm{r}$. Since control charting is an online process we make these calculations after every new observation is observed. This can be tedious and potentially problematic if we have limited computing power. Recursion is one method we can use in order to save computation time and improve calculation efficiency. We present the following equations that update the sample mean Eq. (2-3) and variance Eq. (2-4) iteratively using the findings from Youngs and Cramer (1971).

$$\bar{X}_r = \frac{1}{r}[(r-1)\bar{X}_{r-1} + X_r], \qquad r = 2, 3, \ldots \tag{2-3}$$

$\bar{X}_{r-1}$ is the sample mean observed from the $1 \ldots \mathrm{r} - 1$ samples and its use in the equation above allows us to save computation time. A similar idea is applied to recursively estimate the in-control process variance.

$$S_r{}^2 = \frac{(r-2)}{(r-1)} S_{r-1}{}^2 + \frac{1}{r}(X_r - \bar{X}_{r-1})^2, \quad r = 3, 4, \ldots \tag{2-4}$$

Where $S_{r-1}{}^2$ represents the sample variance calculated from the $1 \ldots \mathrm{r} - 1$ samples.

2.1.2 Q Statistic Derivation

Quesenberry uses these estimates in formulating his *Q* statistics which follow a standard normal distribution. The first case he considers is when the in-control mean and the variance are both known prior to process startup. This first case is only possible when there have been enough observations to safely assume the in-control parameters are known. He proposes the following *Q* statistic in Equation (2-5).

$$Q_r(X_r) = \frac{(X_r - \mu_0)}{\sigma_0} \tag{2-5}$$

In order for the Q statistic to correctly be derived we require the $X_1 \ldots X_r$ to follow normal distribution as stated previously. This will prove critical in the ability to identify the distribution that the Q statistic follows and will allow us to apply any necessary transformations in the later cases. Since the Q chart is a standard normal

Shewhart-type chart we must show that this is in fact a standard normal statistic by checking both the expectation and variance.

$$E(X_r - \mu_0) = E(X_r) - E(\mu_0) = \mu_0 - \mu_0 = 0$$

$$Var(X_r - \mu_0) = Var(X_r) + Var(\mu_0) - 2Cov(X_r, \mu_0)$$

Since μ_0 is known we know the variance is equal to zero so the expression simplifies to

$$Var(X_r - \mu_0) = {\sigma_0}^2$$

We then consider the variance of the statistic as a whole

$$Var\left(\frac{X_r - \mu_0}{\sigma_0}\right) = \frac{1}{{\sigma_0}^2} Var(X_r - \mu_0) = \frac{1}{{\sigma_0}^2}{\sigma_0}^2 = 1$$

Since we were able to show the numerator of the statistic follows normal distribution and the expectation was zero and variance is one we have proven the statistic follows standard normal distribution. We next consider the situation that the mean is unknown but the variance is known. He proposes the following statistic for this situation in Equation (2-6).

$$Q_r(X_r) = \left(\frac{r-1}{r}\right)^{\frac{1}{2}} \frac{(X_r - \bar{X}_{r-1})}{\sigma_0} \tag{2-6}$$

It can be observed that this result is very similar to Equation (2-5) used for the case where both the in-control mean and variance were known. One major difference is that instead of using μ_0 for the case in which the in-control mean is known we use $\bar{X}_{r-1}$ because we need to estimate the in-control process mean. $\bar{X}_{r-1}$ is utilized because it is our best estimate of the in-control sample mean which we base on the $1 \ldots r-1$ observations we have seen prior to the current rth observation. Another major difference that can be observed is the $\left(\frac{r-1}{r}\right)^{\frac{1}{2}}$ term multiplied to the kernel of the statistic. This constant is a byproduct of the uncertainty introduced and is used to ensure the statistic still follows standard normal distribution even though parameter estimation is being used. We again calculate the mean and variance of the statistic to verify it follows a standard normal distribution and to determine the constant that must be applied to preserve this result.

$$E(X_r - \bar{X}_{r-1}) = E(X_r) - E(\bar{X}_{r-1}) = E(X_r) - E(\frac{1}{r-1}\sum_j^{r-1} X_j)$$

$$= E(X_r) - \left(\frac{1}{r-1}\right)E\left(\sum_j^{r-1} X_j\right) = \mu_0 - \left(\frac{1}{r-1}\right)E\left(\sum_j^{r-1} X_j\right)$$

$$= \mu_0 - \left(\frac{1}{r-1}\right)\mu_0(r-1) = 0$$

$$Var(X_r - \bar{X}_{r-1}) = Var(X_r) + Var(\bar{X}_{r-1}) - 2Cov(X_r, \bar{X}_{r-1})$$

The covariance term drops out since $X_r, \bar{X}_{r-1}$ are independent since $\bar{X}_{r-1}$ does not include the observation X_r and therefore their covariance is zero.

$$Var(X_r - \bar{X}_{r-1}) = Var(X_r) + \ Var(\frac{1}{r-1}\sum_j^{r-1} X_j)$$

$$= Var(X_r) + \left(\frac{1}{r-1}\right)^2 Var\left(\sum_j^{r-1} X_j\right)$$

$$Var(X_r) + \left(\frac{1}{r-1}\right)^2 (\sum_j^{r-1} Var(X_j) + \sum_{i \neq j} Cov(X_i, X_j))$$

$$Var(X_r) + \left(\frac{1}{r-1}\right)^2 {\sigma_0}^2(r-1) = {\sigma_0}^2 + \frac{{\sigma_0}^2}{r-1} = \frac{{\sigma_0}^2 r}{r-1}$$

Now considering the variance of the entire statistic we get

$$Var\left(\frac{X_r - \bar{X}_{r-1}}{\sigma_0}\right) = \frac{1}{{\sigma_0}^2}Var(X_r - \bar{X}_{r-1}) = \frac{1}{{\sigma_0}^2}\frac{{\sigma_0}^2 r}{r-1} = \frac{r}{r-1}$$

In order for this statistic to have a variance of one we have to multiply the inverse of $\frac{r}{r-1}$ to arrive at the result for the unknown in-control process mean but known in-control process variance. Then in order to standardize the statistic we divide by the standard deviation so we must take the square root of the expression which gives us $\left(\frac{r-1}{r}\right)^{\frac{1}{2}}$. After determining the constant that allows for preservation of the normally distributed statistic we develop Equation (2-6).

We then consider two cases in which the in-control variance is unknown. The first of which is when the in-control process mean is known but the in-control process variance is not. When the in-control process variance is unknown we must estimate its value using the sample variance equation. The in-control process variance is calculated by using our best estimate of the in-control variance which is the sample variance based on the $1 \ldots r-1$ historical observations seen before the current rth observation. The statistic in Equation (2-7) is used in this situation.

$$Q_r(X_r) = \Phi^{-1}\left\{G_{r-1}\left(\frac{X_r - \mu_0}{S_{0,r-1}}\right)\right\} \qquad r = 2, 3, \ldots \tag{2-7}$$

Where $S^2{}_{0,r} = \frac{1}{r}\sum_{j=1}^{r}(X_j - \mu_0)^2$

Notice here that the statistic has again taken on a different form. In the case of unknown in-control process variance we introduce a distribution transformation G_{r-1} which represents the *t*-distribution function with $r-1$ degrees of freedom. Since the sample variance equation is used in the denominator of our statistic when the variance is unknown it changes the distribution of the statistic. The numerator still follows a normal distribution but the denominator is no longer a constant and is now a random variable since the in-control process variance is unknown. Since we utilize the sample variance equation that is derived from the summation of independent squared normal random variables we find that the denominator follows chi-square distribution. Because of this numerator and denominator combination we suspect that our statistic follows a *t*-distribution. We find that this is in fact the case and we apply the G_{r-1} transformation which turns our *t*-distributed random variable with $r-1$ degrees of freedom into something we can transform into a standard normal statistic with the use of Φ^{-1} the inverse of the standard normal distribution function. We will show these distribution results more closely in our discussion of Case IV.

We now consider the most relevant case to self-starting applications, the case when both the in-control process mean and in-control process variance are unknown. This is the most common situation since in the case of short runs or data collection limitations a prime motivation for self-starting charts is to solve the problem of not knowing the process parameters prior to charting. Equation (2-8) is the statistic proposed for the situation in which both the in-control mean and variance are unknown. Because of this, the statistic must utilize both the constant and the distribution transformation in order to ensure the statistic follows standard normal distribution.

$$Q_r(X_r) = \Phi^{-1}\left\{G_{r-2}\left[\left(\frac{r-1}{r}\right)^{\frac{1}{2}}\left(\frac{X_r - \bar{X}_{r-1}}{S_{0,r-1}}\right)\right]\right\} \qquad r = 3, 4, \ldots \tag{2-8}$$

We now show the derivation of this statistic in detail. Quesenberry suspects that $\left(\frac{X_r - \bar{X}_{r-1}}{S_{0,r-1}}\right)$ follows a *t*-distribution and writes the statistic above in a different fashion in order to more easily identify its distribution. In order for this statistic to follow a *t*-

distribution with ν degrees of freedom it must first take on the form of $T = \frac{Z}{\sqrt{\frac{U}{\nu}}}$ where Z is a standard normal random variable, and U follows a chi-square distribution. In addition, Z and U must be independent. We can express the statistic in the following way to show it follows a *t*-distribution.

$$Z = \left(\frac{r-1}{r}\right)^{\frac{1}{2}} \left(\frac{X_r - \bar{X}_{r-1}}{\sigma_r}\right)$$

$$U = \frac{(r-2){S_{r-1}}^2}{{\sigma_r}^2}$$

$$\nu = r - 2$$

Which takes on the same form as before when written in this way.

$$T = \frac{\left(\frac{r-1}{r}\right)^{\frac{1}{2}} \left(\frac{X_r - \bar{X}_{r-1}}{\sigma_r}\right)}{\sqrt{\frac{(r-2){S_{r-1}}^2}{(r-2){\sigma_r}^2}}} = \left(\frac{r-1}{r}\right)^{\frac{1}{2}} \left(\frac{X_r - \bar{X}_{r-1}}{S_{r-1}}\right)$$

We proved previously that Z follows a standard normal distribution with expectation zero and variance one. We must now show that U follows a chi-square distribution with ν degrees of freedom. For a chi-square distribution we need $Z_1, \dots, Z_k$ independent standard normal random variables. Then we can write the sum of squares in this way.

$$Q = \sum_{i=1}^{K} {Z_i}^2 \qquad\qquad Q \sim {\chi_k}^2$$

We then consider the sample variance Equation (2-2) and the best estimate of the variance at time r which is the sample variance at time $r - 1$.

Quesenberry (1991) uses the best estimate of the sample variance in the case of unknown variance which is based on the $1,2, \dots r-1$ observations.

$$S_{r-1}{}^2 = \frac{1}{r-2}\sum_{j=1}^{r-1}(X_j - \bar{X}_{r-1})^2 \tag{2-9}$$

A commonly used result in statistics tells us that the sample mean is independent of the sample variance. Since $S_{r-1}{}^2$ is composed of the sum of independent standard normal random variables it can be concluded that it follows chi square distribution with $r-2$ degrees of freedom. After verifying that the statistic developed thus far follows a *t*-distribution we now know the appropriate transformation to apply in order to ensure the statistic follows standard normal distribution. In order to get the statistic in a form that can be transformed to standard normal we apply the student *t*-distribution function which will transform the statistic into a probability value. Now that this statistic is simplified into simply a probability value we can apply the inverse of the standard normal distribution function to put the statistic in terms of a quantile of the standard normal distribution.

2.1.3 Independence among Q Statistics

A desirable property of charting statistics in SPC is independence among observations. Ideal charting performance and accuracy can be reached when the independence of statistics can be attained. Quesenberry was able to show the independence of his *Q* statistic between observations. He shows this result through the use of two lemmas in the proof presented below. Let Y_1 and Y_2 be independent chi–square random variables with degrees of freedom ν_1 and ν_2 respectively. Then we can conclude that the ratio Y_1/Y_2 and the sum $Y_1 + Y_2$ are independent random variables.

The second lemma shown states that if Y_i for $i = 1,2,3$ are independent chi-square random variables with ν_i degrees of freedom then

$$W_1 = \frac{\nu_2 Y_1}{\nu_1 Y_2}$$

$$W_2 = Y_1 + Y_2$$

$$W_3 = \frac{\nu_1 + \nu_2}{\nu_3} \frac{Y_3}{Y_1 + Y_2}$$

Are independent F_{ν_1,ν_2} and $F_{\nu_3,\nu_1+\nu_2}$ random variables where F_{ν_1,ν_2} represents the F-distribution function with numerator degrees of freedom ν_1 and denominator degrees of freedom ν_2. Using conclusions made from Lemma 1 it is easily shown that W_1 and W_2 are independent F random variables. W_1 is independent of $Y_1 + Y_2$ and since W_1 is also independent of Y_3 it is independent of any function of Y_3 and $Y_1 + Y_2$.

Using the two lemmas proposed Quesenberry (1991) defines the following quantities.

$$Y_1 = \frac{1}{\sigma^2} \sum_{i=1}^{r-1} (X_i - \bar{X}_{r-1})^2$$

$$Y_2 = \left(\frac{r-1}{r}\right) \frac{(X_r - \bar{X}_{r-1})^2}{\sigma^2}$$

$Y_3 = \left(\frac{r}{r+1}\right) \frac{(X_{r+1} - \bar{X}_r)^2}{\sigma^2}$ (1991) shows the independence of Y_1 and Y_2

$$Cov(X_i - \bar{X}_{r-1}, X_r - \bar{X}_{r-1})$$

$$= E[(X_i - \bar{X}_{r-1})(X_r - \bar{X}_{r-1})]$$

$$= E(\bar{X}_{r-1} - \mu^2) - E[(X_i - \mu)(\bar{X}_{r-1} - \mu)]$$

$$= \frac{\sigma^2}{r-1} - \frac{\sigma^2}{r-1} = 0$$

Since $(X_i - \bar{X}_{r-1})$ and $(X_r - \bar{X}_{r-1})$ are normal, zero covariance proves independence. The other pairs can be shown to be independent following similar logic.

$$A_r{}^2 = \frac{(r-2)Y_2}{Y_1} = \left(\frac{r-1}{r}\right)\frac{(X_r - \bar{X}_{r-1})^2}{S_{r-1}{}^2}$$

$$A_{r+1}{}^2 = \frac{(r-1)Y_3}{Y_1 + Y_2} = \left(\frac{r}{r+1}\right)\frac{(X_{r+1} - \bar{X}_r)^2}{S_r{}^2}$$

Quesenberry (1991) continues his application of the second Lemma presented earlier to show that A_r and A_{r+1} are independent. It can also be seen that A_r and A_{r+1} follow *t*-distribution so therefore Q_r and Q_{r+1} are independent standard normal random variables from the probability integral transformation. It is important to notice the different ranges for the *Q* statistic corresponding to each of the different cases. In case I both of the in-control parameters are known so the estimates do not have to be used. Since these are both known the Q_1 for X_1 can be obtained. However, when looking at cases II and III it can be seen that because of estimation of the parameters the range of the statistic becomes limited to $r = 2,3$ ... so the Q_1 for X_1 cannot be calculated. In case four when we have two unknown parameters the range becomes $r = 3,4$ ... because of estimation meaning we cannot calculate values for Q_1 for X_1 or Q_2 for X_2. This result can be generalized. If p represents the number of unknown parameters then we obtain the *Q* statistics starting with Q_{p+1}.

After the *Q* statistic values have been calculated they can then be plotted on a Shewhart control chart. The Shewhart control chart will utilize control limits at ±3 since our statistic follows a standard normal distribution.

2.2 Linear Regression Model

In order to solve our problem we model the in-control state of the process as a linear trend rather than the traditional stable average. We model this non-stable mean using a linear least squares simple regression model. We now offer an explanation of the basic concepts of linear regression.

For our application we consider a linear regression model with a single predictor which is commonly known as simple linear regression. We have a pair (x_i,y_i) for each $1,2 \ldots n$ observation and we model the relationship of these n observations in Equation (2-10).

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \qquad i = 1,2, \ldots, n \tag{2-10}$$

Where β_0 is the intercept parameter and β_1 represents the slope parameter. For standard linear regression analysis we have assumptions about the explanatory term and the error term ε_i. The first regards the regressor variable x_i. The regressor variable is under the experimenter's control so the $x_i's$ are not random variables and can be taken as constants. Secondly, the error term has expectation zero which implies the expectation of the response variable is $\beta_0 + \beta_1 x_i$. Another assumption regarding the error term is that its variance is constant meaning that the variance of the response variable at any observation is the same as at all the other times. Lastly, the errors at each time are independent of one another which also implies this same independence amongst the response variables at any given time observation. The ultimate goal of linear regression is to establish a relationship between the response and the explanatory variable that will give us predicting ability for future observations which we will be helpful to us in solving our problem.

2.2.1 Regression Parameter Estimation

The regression parameters β_0 and β_1 hold much of the predictive power in our linear model. β_0 is the intercept parameter and β_1 represents the slope parameter. These parameters are often not known in practice and therefore must be estimated. There are two common ways that parameters are estimated in linear regression, maximum likelihood estimation and least squares estimation. The maximum likelihood estimation selects parameters based on maximizing the likelihood function for the parameters. However, we will focus more heavily on the least squares estimation in this work. The least squares method uses the squared distance as a measure of closeness and minimizes the function below to estimate the intercept and slope parameters.

$$S(\beta_0, \beta_1) = \sum (y_i - \beta_0 - \beta_1 x_i)^2 \tag{2-11}$$

From taking the derivative of this function with respect to β_0 and β_1 and setting it equal to zero we obtain the equations to estimate β_0 and β_1.

$$\hat{\beta}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} \tag{2-12}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \tag{2-13}$$

We pay close attention to the residual in later results. The residual is the vertical distance between the observation y_i and the estimated line evaluated at x_i. Another useful least squares estimation is the mean square error (MSE) which is the unbiased least squares estimate of the standard deviation.

$$MSE = s^2 = \frac{S(\hat{\beta}_0, \hat{\beta}_1)}{n - 2} \tag{2-14}$$

2.2.2 Distributions of the Parameter Estimates

Since we will be constructing a statistic involving the parameter estimates it is important to establish the distributions of these random variables. We consider the distribution of the following quantities to aid in our future analysis. $\hat{\beta}_1$ is an unbiased estimate of β_1 and has been found to follow a normal distribution $\hat{\beta}_1 \sim N(\beta_1, \sigma^2/s_{xx})$. Because $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$ is a linear combination of normal random variables it to is normally distributed $\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{s_{xx}}\right]\right)$.

In order to monitor our in-control linear trend we wish to represent the trend in terms of a single value to preserve simplicity and interpretability in our statistic. This can be accomplished by writing the relationship in terms of the estimated response. We define the quantity $\hat{\mu}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$ which is the estimated value of the response variable at x_0 based on the least squares parameter estimates of the regression parameters. We consider the mean and variance for $\hat{\mu}_0$.

$$E(\hat{\mu}_0) = \beta_0 + \beta_1 x_0 = \mu_0$$

Therefore $\hat{\mu}_0$ is an unbiased estimator for μ_0. We then look at the variance of $\hat{\mu}_0$.

$$Var(\hat{\mu}_0) = \sigma^2\left[\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{s_{xx}}\right]$$

Where s_{xx} is the sum of squares x and is calculated in the following way.

$$s_{xx} = \sum (x_i - \bar{x})^2$$

From these results we can say that $\hat{\mu}_0 \sim N\left(\mu_0, \sigma^2\left[\frac{1}{n} + \frac{(x_0-\bar{x})^2}{s_{xx}}\right]\right)$.

In this work we will also use the vector interpretation of the linear regression results presented above. Using matrix operations can often times be easier than dealing with all scalar quantities. In addition to its potential ease of use the vector notation of the regression model is necessary when dealing with multiple regression which is the situation in which we want to predict the value of a variable based on two or more regressor variables. We can then model our regression relationship using vectors as shown in Equations (2-15) through (2-17).

$$y = X\beta + \varepsilon \tag{2-15}$$

X is what is known as the design matrix in which the first column is a column of ones for the intercept term in the model. β is the vector of regression parameter estimates and ε is once again the error term. The β matrix can be estimated using the following matrix operations which results in the least squares estimate of β.

$$\hat{\beta} = (X'X)^{-1}X'y \tag{2-16}$$

$\hat{\beta}$ is an unbiased estimator of β and the variance of the estimated parameter matrix is shown below.

$$Var(\hat{\beta}) = (X'X)^{-1}\sigma^2 \tag{2-17}$$

CHAPTER 3 PROPOSED METHODOLOGY

3.1 Problem Formulation

Let $Y_1, Y_2 \ldots Y_t$ represent measurements collected in time sequence that are independently and identically distributed having been collected from a normal distribution. We will utilize the information from these observations to help us in estimating regression parameters to monitor a process that follows an in-control linear trend using a Q chart methodology. In order to set the stage for our Q chart we consider the explanatory variable X_t which is the value of the independent variable at time t. Correspondingly, Y_t is the response variable in our linear model at time t. We use the model developed from simple linear regression to describe the relationship between our independent and dependent variable in equation (3-1).

$$Y_t = \beta_0 + \beta_1 X_t + \epsilon_t \tag{3-1}$$

where

$\epsilon_t \sim N(0, \sigma^2)$

In this thesis, $X_t = t$ so that Eq. (3-1) is corresponding to a linear trend. The above regression equation is used to represent the in-control state of the degradation process. β_0 represents the y-intercept of the model and β_1 represents the slope. However, we frequently do not know the value of the in-control parameters prior to starting the run so we must estimate the regression parameters β_0, β_1, and σ. The method we will use for parameter estimation in our linear regression is least squares estimation. The following equations Eq. (3-2) and Eq. (3-3) allow for calculation of the least squares estimates of β_0 and β_1 at time t.

$$\hat{\beta}_1^{(t)} = \frac{\sum_{i=1}^{t}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{t}(X_i - \bar{X})^2} \tag{3-2}$$

$$\hat{\beta}_0^{(t)} = \bar{Y} - \hat{\beta}_1\bar{X} \tag{3-3}$$

These estimates can then be used to predict the value of the observations Y_t at time t, which is given by

$\hat{Y}_t = \hat{\beta}_0^{(t)} + \hat{\beta}_1^{(t)} X_t$ where t=1,2 ...

We also wish to estimate the in-control variance when the value of the parameters is not known prior to starting the run. This estimation is based on the mean squared error (MSE) in linear regression analysis and is shown below in Eq. (3-4).

$$MSE_t = s_t^2 = \frac{S(\hat{\beta}_0^{(t)}, \hat{\beta}_1^{(t)})}{t-2} \tag{3-4}$$

Where

$$S(\hat{\beta}_0^{(t)}, \hat{\beta}_1^{(t)}) = \sum_{i=1}^{t} (Y_i - \hat{\beta}_0^{(t)} - \hat{\beta}_1^{(t)} X_i)^2 \tag{3-5}$$

3.2 Extension of the *Q* Statistic to Linear Trend Data

We showed in chapter 2 that, if the in-control data have a constant mean, Quesenberry (1991) develops his *Q* statistic at current time *t* by subtracting the best estimate of the in-control process mean and dividing that quantity by the best estimate of the standard deviation based on the previous observations. We will follow a similar idea to extend the *Q* statistics for in-control state with linear trends. In our model, $\hat{Y}_{t-1} = \hat{\beta}_0^{(t-1)} + \hat{\beta}_1^{(t-1)} X_t$ will be used to replace the estimate of the in-control process mean and s_{t-1} is our best estimate of the in-control standard deviation when neither parameter is known prior to starting the run.

For self-starting charts the idea of using a predicted value in our statistic stems from the thought that the current value, which may be out of control, potentially contaminates our estimate of the in-control state. We hypothesize that if the performance of the chart improves when a predicted value based on the $t-1$ observations is used that the effects of contamination will be even further reduced if we introduce a greater delay. Up to this point we have discussed the case where the predicted value of Y_t is based on the $1,\ldots\, t-1$ previous observations, $\hat{Y}_{t-1}$. Following this idea we would like to introduce some new notation. The observations up to time *t-d* are used to estimate the in-control parameters and predict the observation at time *t*, Y_t, which is the *d*-step ahead prediction at time *t-d*, denoted by $\hat{Y}_{t-d}(d)$. It is easy to see

$$\hat{Y}_{t-d}(d) = \hat{\beta}_0^{(t-d)} + \hat{\beta}_1^{(t-d)} X_t \tag{3-6}$$

The parameter $d \geq 1$ is called *delay parameter* in this thesis and the corresponding *Q* statistics are called *Q* statistics with delay *d*.

The first case we visit is when the in-control process mean is unknown but the in-control process variance is known. In this case we only have to estimate the in-control process mean and the in-control process variance is considered a known constant. Like we showed in Eq. (2-6) a constant must be multiplied to our statistic in order to preserve the standard normal distribution of the *Q* statistic. Below we show our statistic for our first case with in-control process mean unknown and in-control process variance known that incorporates the aforementioned constant.

3.2.1 The Case of Unknown in-control Mean and Known Standard Deviation

Case I: μ unknown and σ known

When the in-control variance is known, we propose the following extended Q statistics with delay d at time t:

$$Q_t(Y_t;d) = \frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{\sigma} \qquad (3\text{-}7)$$

for $t = 2 + d, 2 + d + 1, \ldots$

We will now show the distribution of the statistic in Eq. (3-7) follows a standard normal distribution. That is, $Q_t(Y_t;d) \sim N(0,1\,)$. We begin the derivation of the Case I statistic with the following kernel.

$$\frac{Y_t - \hat{Y}_{t-d}(d)}{\sigma}$$

The ultimate goal of our quality statistic is to have the ability to chart our in-control linear trend on a standardized Shewhart Chart by using Q chart methodologies. In order to show this expression can be transformed into a standardized normal Q statistic we must first show that it has zero mean and $\sigma^2 = 1$. This is shown here.

$$E\left(Y_t - \hat{Y}_{t-d}(d)\right) = E(Y_t) - E(\hat{Y}_{t-d}(d))$$
$$= E(\beta_0 + \beta_1 X_t + \epsilon_t) - E(\hat{\beta}_0{}^{(t-d)} + \hat{\beta}_1{}^{(t-d)} X_t)$$
$$= \beta_0 + \beta_1 X_t - \beta_0 + \beta_1 X_t = 0$$

Since the numerator is zero we have proven that the mean of the statistic is zero. We then calculate the variance of the numerator to verify that it in fact equals one.

$$V\left(Y_t - \hat{Y}_{t-d}(d)\right) = V(Y_t) + V\left(\hat{Y}_{t-d}(d)\right) - 2cov(Y_t, \hat{Y}_{t-d}(d))$$

It is easy to see $cov(Y_t, \hat{Y}_{t-d}(d)) = 0$, since Y_t is the response at time t and $\hat{Y}_{t-d}(d)$ is the estimated response at t which only includes time up to time $t-d$ and not Y_t. Therefore, this means that Y_t is independent of $\hat{Y}_{t-d}(d)$ and that $cov(Y_t, \hat{Y}_{t-d}(d)) = 0$.

$$V\left(Y_t - \hat{Y}_{t-d}(d)\right) = V(Y_t) + V(\hat{\beta}_0{}^{(t-d)} + \hat{\beta}_1{}^{(t-d)} X_t)$$
$$= \sigma^2 + V(\bar{y}_{t-1} - \hat{\beta}_1{}^{(t-d)} \bar{X}_{t-d} + \hat{\beta}_1{}^{(t-d)} X_t)$$
$$= \sigma^2 + V(\bar{y}_{t-1} - \hat{\beta}_1{}^{(t-1)} (X_t - \bar{X}_{t-1}))$$
$$\hat{\beta}_1{}^{(t-d)} = \frac{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})y_i}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}$$

We substitute $\hat{\beta}_1{}^{(t-d)} = \frac{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})y_i}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}$ into the equation and factor the summation out front in order to simplify our calculation of the variance.

$$V\left(Y_t - \hat{Y}_{t-d}(d)\right) = \sigma^2 + V\left(\sum_{i=1}^{t-d}\{\frac{y_i}{t-d} + (X_t - \bar{X}_{t-d})\frac{(X_i - \bar{X}_{t-d})y_i}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\}\right)$$
$$= \sigma^2 + V\left(\sum_{i=1}^{t-d}\{\frac{1}{t-d} + \frac{(X_i - \bar{X}_{t-d})(X_t - \bar{X}_{t-d})}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\}y_i\right)$$

We then define

$$C_i = \left(\frac{1}{t-d} + \frac{(X_i - \bar{X}_{t-d})(X_t - \bar{X}_{t-d})}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right)$$

$$V\left(Y_t - \hat{Y}_{t-d}(d)\right) = \sigma^2 + V(\textstyle\sum_{i=1}^{t-d} C_i\, y_i)$$
$$V\left(Y_t - \hat{Y}_{t-d}(d)\right) = \sigma^2 + \sum_{i=1}^{t-1} C_i{}^2\, \sigma^2$$

$$= \sigma^2\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]$$

A full derivation of this is included in the appendix of this thesis. In order to standardize our statistic we must divide by the standard deviation. Following this thought we take the square root of the quantity shown above to get the following.

$$S.D.\left(Y_t - \hat{Y}_{t-d}(d)\right) = \sigma\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}$$

We can then use this result in our statistic to form a standardized Q statistic. As mentioned before we must multiply the statistic by a constant to ensure that it follows standard normal distribution. We multiply by the reciprocal of the constant above for this reason.

$$Z = \frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{\sigma}$$

Now we must verify that this is a standard normal statistic. So far we know it is normally distributed and that the expectation is zero from our previous results. We must check the variance of the statistic to verify that it in fact has variance one.

$$V(Z) = V\left(\frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{\sigma}\right)$$

$$= \frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]} \frac{1}{\sigma^2} V\big(\, Y_t - \hat{Y}_{t-d}(d)\big)$$

By substituting the value of $V\big(\, Y_t - \hat{Y}_{t-d}(d)\big)$ found earlier

$$V(Z) = \frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]} \frac{1}{\sigma^2} * \sigma^2 \left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right] = 1$$

So since $V(Z) = 1$ and $E(Z) = 0$ this is a standard normal statistic and we have verified that the statistic proposed in case I is in fact a standardized Q statistic.

3.2.2 The Case of Unknown in-control Mean and Unknown Standard Deviation

We then extend our analysis of case I to case II: μ and σ^2 unknown. For case II we must estimate the in-control process variance in addition to the in-control process mean which will make it more difficult to transform our statistic into a standard normal *Q* statistic. In case II since σ^2 is unknown we must estimate this value. The traditional method for estimating σ^2 in simple linear regression is to calculate the mean square error (MSE) s^2 by taking the residual sum of squares and dividing by the error degrees of freedom. Below is the equation for the MSE (s^2) that we have generalized for our delay parameter *d*.

$$MSE_{t-d} = s_{t-d}^2 = \frac{S(\hat{\beta}_0{}^{(t-d)}, \hat{\beta}_1{}^{(t-d)})}{t-2-d} \tag{3-8}$$

Where

$$S(\hat{\beta}_0{}^{(t-d)}, \hat{\beta}_1{}^{(t-d)}) = \sum_{i=1}^{t-d} (y_i - \hat{\beta}_0{}^{(t-d)} - \hat{\beta}_1{}^{(t-d)} x_i)^2 \tag{3-9}$$

We now have established the in-control parameters and we must estimate and propose the following statistic for our case II.

Case II: μ and σ^2 unknown

$$Q_t(Y_t; d) = \Phi^{-1}\left\{ G_{t-2-d}\left[\frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{s_{t-d}} \right] \right\} \tag{3-10}$$

for $t = 3 + d, 3 + d + 1 \ldots$

We note that in the above statistic that the prediction of Y_t with a delay of *d*, $\hat{Y}_{t-d}(d)$, is calculated based on observations 1 through $t - d$ (not including time *t*). s_{t-d}

is our estimate of the standard deviation based on the first $t - d$ observations. Since in case II we do not know the value of the in-control process variance our best estimate of σ^2 becomes ${s_{t-d}}^2$ and therefore we divide by its square root to standardize our statistic. Notice that in addition to multiplying by the same constant derived for Case I we also have a distribution transformation. This distribution transformation becomes necessary when s_{t-d} is used because it makes the inner quantity follow *t*-distribution like what was discussed in chapter 2. G_{t-2-d} represents the student *t*-distribution function with $\nu = t - 2 - d$ degrees of freedom and is used here to evaluate our *t*-distributed statistic as a probability value. After doing this we still need our statistic to be expressed in a form we can plot on a standard normal Shewhart-type control chart. We accomplish this through also utilizing the inverse of the standard normal distribution, Φ^{-1}, which transforms our statistic into a standard normal quantile value. We will now offer a more detailed look into our proposed statistic for Case II: unknown in-control process mean and variance.

We apply a similar idea to formulate our statistic in Case II as we did for Case I. We begin with our statistic from case I but make one substitution. Since we do not know the value of the in-control process standard deviation we replace σ with s_{t-d}.

$$\left[\frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-1}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{s_{t-d}}\right]$$

Because of the general form this expression takes we suspect it may follow a *t*-distribution since the denominator has been proven to follow chi-square. So we will try to prove that it in fact follows *t*-distribution. We then represent its components in the following way for ease in identifying its distribution. Define

$$Z = \frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{\sigma}$$

$$U = \frac{(t-2-d){s_{t-d}}^2}{\sigma^2}$$

$$\nu = t-2-d$$

This can be written as

$$T = \frac{Z}{\sqrt{\frac{U}{\nu}}} = \frac{\frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{\sigma}}{\sqrt{\frac{(t-2-d){s_{t-d}}^2}{\sigma^2(t-2-d)}}}$$

$$= \left[\frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{s_{t-d}}\right]$$

For the statistic above to follow a *t*-distribution we must verify properties of *Z*, *U*, and ν. The first condition is that *Z* must follow standard normal distribution with mean zero and standard deviation one which we have verified earlier in this work. The next condition we need for our statistic is for *U* to follow a chi-square distribution. We take a closer look at what makes up *U*.

$${s_{t-d}}^2 = \frac{\sum_{i=1}^{t-d}(y_i - {\hat{\beta}_0}^{(t-d)} - {\hat{\beta}_1}^{(t-d)} X_i)^2}{t-2-d}$$

$$U = \frac{(t-2-d){s_{t-d}}^2}{\sigma^2}$$

It has been proven by Abraham and Ledolter (2006, 132) that the least squares estimates are independent of the residual sum of squares of the least squares estimates. And it further proves that the residual sum of squares divided by σ^2 follows ${\chi^2}_{t-p-d} = {\chi^2}_{t-2-d}$ where *p* is the number of parameters after the intercept.

The last condition we need to show to prove this statistic follows a *t*-distribution is that *Z* and *U* are independent. We can arrive at this result through considering what we know about the individual terms that make up the *Z* and *U*. Notice that *U* does not contain the Y_t observation, it only contains up to the observation at time $t-d$ which means Y_t and *U* are independent. We must also verify that $\hat{Y}_{t-d}(d)$ is independent of *U*. $\hat{Y}_{t-d}(d) = {\hat{\beta}_0}^{(t-d)} + {\hat{\beta}_1}^{(t-d)} X_t$ so is composed of estimates of beta. Abraham and Ledolter (2006, 132) verifies that the parameter estimates and s^2 are independent which means that $\hat{Y}_{t-d}(d)$ is independent of *U*.

After proving all the conditions above we can say.

$$\left[\frac{1}{\left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]\right]^{\frac{1}{2}}} \frac{Y_t - \hat{Y}_{t-d}(d)}{s_{t-d}}\right] \sim G_{t-2-d}$$

We note here that for both Case 1 Eq. (3-7) and Case 2 Eq. (3-10) that when the delay parameter *d*=1 Q_t statistics are independent of each other i.e. Q_t is independent of Q_{t-1}. This result is shown in (Koning, 1997).

3.3 The EWMA Charting Scheme

As discussed previously a useful charting technique is the exponentially weighted moving average (*EWMA*) control chart. The *EWMA* chart assigns weights to the previously observed points in order to represent the process's history. The *EWMA* chart is set up so it assigns the largest weight to the most recent observation and assigns an exponentially decreasing weight to each subsequent older observation. The parameter that must be established by the engineer is λ which is commonly called the smoothing constant. The smoothing constant, λ, is established as the weight given to the most recent rational subgroup mean and the statistic decays in value from there with time. The weights are set up so they sum to unity and the oldest observation is dropped from the weighted average as a new one is observed and added to the weighted average.

We expect that applying a *EWMA* scheme will further improve our charting technique because while still reducing the contamination effect using the delay *d* parameter in our statistic we can also improve the charts' signal detection speed after a shift. By weighting the influence of the most recent observations higher than the older observations we are able to detect smaller shifts and detect them faster.

We now present the application of *EWMA* to our *Q* statistic. The exponentially weighted moving average is defined in Eq. (3-11).

$$Z_t(d) = \lambda * Q_t(Y_t; d) + (1 - \lambda)Z_{t-1}(d) \tag{3-11}$$

where $0 < \lambda \leq 1$.

After applying a *EWMA* to our $Q_t(Y_t; d)$ statistic we will chart the value $Z_t(d)$ in Eq. (3-11) for each time 1…*t-d* as it is observed. Note that the starting value corresponds to the process target. We intend on charting on a standardized normal control chart which means our centerline will be at zero since $Q_t(Y_t; d) \sim N(0,1)$. So in our case since the Center Line is zero we define $Z_0 = 0$. We define the corresponding control limits and center line of our *EWMA* chart in Eq. (3-12) and Eq. (3-13).

$$UCL = L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2t}]} \tag{3-12}$$

$$LCL = -L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2t}]} \tag{3-13}$$

We note here that the control limits look very different from what was presented earlier in the thesis for *Q* charts. The difference here stems from the application of the *EWMA*. *Q* charts use constant Shewhart chart control limits but because of the $[1-(1-\lambda)^{2t}]$ term used in *EWMA* we get non-constant control limits. This term however approaches unity as *t* grows larger meaning the control limits will approach steady state values after the chart has been running for several time periods.

3.4 Simulation Study

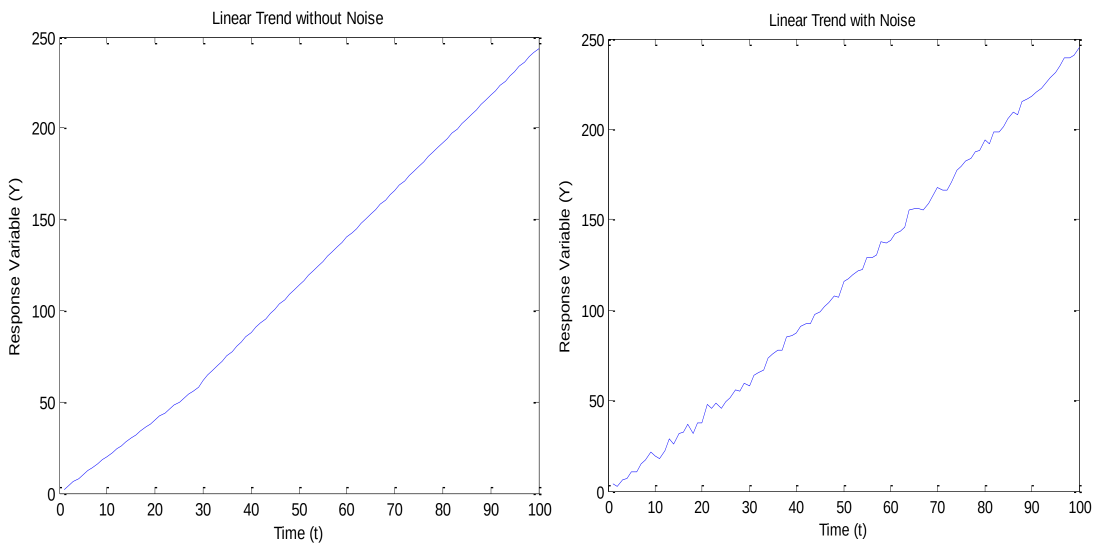
In this section we will discuss our results and conclusions from the testing of our method through simulation examples. We offer an evaluation of charting performance based on Average Run Length values (ARL) calculated through simulation study.

ARL is a common measure used as an evaluator of control charting performance. There are two main ARL measures considered. The in-control ARL is defined as the average number of data points that must be plotted before a point indicates an out of control condition when the process is in-control, so the higher the in-control ARL the better the performance of the chart. The in-control ARL can be thought of as the chart's susceptibility to signaling the process is out of control when it is in fact in-control. We desire this in-control ARL to be as high as possible and we compare our charting performance to that of the Shewhart chart with $\pm 3\sigma$ control limits. The Shewhart Chart assumes independent normal distributed data so we determine the expected in-control ARL to be the inverse of the probability that a charting statistic exceeds $\pm 3\sigma$. The other type of ARL, the out of control ARL, or ARL_1, is defined as the average amount of samples needed to detect a process shift when the process is in the out of control state. Both ARL should be considered when an engineer designs a control chart. It is desirable to have a short out of control ARL and a long in-control ARL.

In designing a charting method, typically the in-control ARL is fixed at a certain value, effectively fixing the false alarm rate, so charts can be evaluated and compared based solely on their out of control ARL performance. For example the expected in-control ARL for a Shewhart-type chart with $\pm 3\sigma$ control limits is 370. So in testing we wish to adjust the control limits of all the other charting methods we are comparing to the Shewhart chart so their in-control ARLs are approximately 370. All other things equal a shorter out of control ARL indicates better charting performance since this measure refers to the average amount of observations seen after a shift needed before the chart detects it.

We simulate independent normally distributed data following a certain in-control linear trend using Matlab. At a certain time which we will call the change-point, τ, we will introduce a shift in either the slope or the intercept parameter and observe how quickly and accurately our proposed Q chart based methods respond. We plan to simulate several scenarios varying slope shift, intercept shift and the number of in-control points observed prior to the shift with 2000 iterations for each test.

We now consider a shift in slope that closely simulates the situation in which we would apply our method in practice. Small linear trend shift detection is made possible with our proposed methodology so a shift such as in Figure 2 below that is 1.5 times the original slope parameter can be detected accurately and efficiently. As can be seen this shift is hard to detect when only considering a visual inspection. Figure 2 shows the mean of the observations with a change point at time 30.



(a) (b)

Figure 2: Simulated Linear Trend with Slope Shift at Change Point 30

The mean curve in Figure 2 can be generated from the following equation:

$$Y_t = \begin{cases} Y_t = \beta_0 + \beta_1 X_t & for\ t = 1,2 \ldots 1 - \tau \\ Y_t = \beta_0{}' + \beta_1{}' X_t & for\ t = \tau, \tau + 1, \ldots \end{cases} \tag{3-14}$$

where

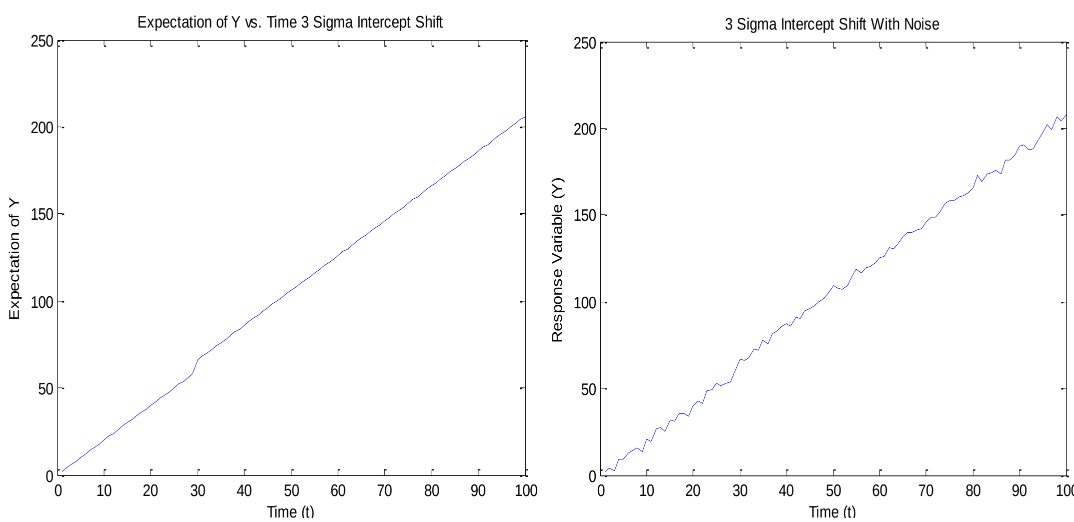
$\beta_1' = \gamma\beta_1$

$\beta_0' = \beta_1 X_\tau + \beta_0 - \beta_1' X_\tau + \alpha$

τ is the changepoint, α is the intercept shift and γ is the slope shift factor

Equation (3-14) shows how we define our in-control linear trend and how the out of control linear trend is related to the in-control state. We can see from Figure 2 above that the linear trends we consider are continuous functions that meet at a common point where the regression parameters change. This meeting point between the linear trends is referred to as the change point and is denoted by τ. In Figure 2(a) we show the observations that are subject to a slope shift at change point 30. In practice, the observations are noisy as in the model given in Equation (3-1). In Figure 2(b) we show the observations with a random error term that simulates noise. Once the random error term is included it becomes nearly impossible to identify where the change point occurs or if a slope or intercept shift even happened at all through only visual inspection. It is for this reason that we propose using an SPC method to detect this shift.

In this simulation study we also consider intercept shifts. We show below the intercept shifts we tested our charting methods on. Like in Figure 2 we show the observations that are subject a change in the parameter values at change point 30. In Figure 3(b) we show the observations with a random error term that simulates noise but this time with an intercept shift. In Figure 3 we show a three sigma magnitude y-intercept shift.



(a) (b)

Figure 3: Simulated Linear Trend with Intercept Shift at Change Point 30

It is clear to see that this is once again a very slight shift in the original linear trend. This becomes extremely hard to detect through visual inspection once the noise term is considered.

In this thesis we will study and compare the three methods we proposed: the Shewhart self-starting *Q* chart for linear trend (referred to as Shewhart in this thesis), the EWMA chart for Q statistics (referred to as EWMA in this thesis), and the *EWMA* with delay *d* (referred to as EWMA-*d* in this thesis). The first step we need to take in comparing ARL performance is to make sure that the in-control ARL for all the charts is approximately the same. This step is necessary because without fixing the in-control ARL we could not fairly compare the methods. We use the established in-control ARL for the Shewhart charts with three sigma control limits of 370 and find a control limit value of the other charts so we can compare them. We referenced control limit values for the *EWMA* chart to match with those of the Shewhart chart and estimated the necessary control limits of our chart through simulation. For the Shewhart charts based on *Q*-statistics, as we showed in Section 3.2 that the *Q*-statistics extended for linear trends are i.i.d. standard normal random variables, analytical approach is available to find the control limit for any given ARL. For *EWMA* based control charts with or without delays, we must determine the proper control limits through Monte Carlo simulation. We use Monte Carlo simulation to evaluate the in-control ARL for any given control limit and adjust the control limits so that the corresponding ARL is close enough to the desired value. We show the control limits for the desired in-control ARL for the three charts in Table 1 below.

Table 1: In-Control ARL Simulation Estimates

Weight (λ)	d step Parameter (d)	Control Limit (L)	Chart Type	ARL_0
1.0	1.0	3.00	Shewhart	370.4
0.2	1.0	2.86	EWMA	370.5
0.2	2.0	2.9339	EWMA-*d*	370.6
0.2	3.0	2.988	EWMA-*d*	370.9
0.2	4.0	3.0185	EWMA-*d*	370.5
0.2	5.0	3.0358	EWMA-*d*	370.7

We tested several different scenarios in order to compare the performance of our charting schemes. We tested varying degrees of both slope and intercept shifts. We show in Table 2 the slope, intercept, and standard deviation we used in our simulation study. The results are shown in Tables 3, 4, 5, and 6. We again use the notation from Eq. (3-14).

Table 2: Parameter Values used in Simulation Study

Parameter	Value
Slope (β_1)	2.0
Intercept (β_0)	0.0
Standard Deviation (σ)	4.0

We first tested our chart schemes for what we designated as Case 1 which corresponds to the case where the in-control standard deviation is known but the in-control variance is not. We varied both the slope factor and the intercept shift factor and ran simulations to observe the out of control ARL. Our results for slope and intercept shifts with 30 in-control data points can be seen in Table 3. We vary the slope shift factor from 1.5 to 2.5 and included intercept shifts of twice and three times the standard deviation.

Table 3: Case 1 Out of Control ARL Simulation Results In-control History of 30 points

Weight (λ)	delay Parameter (d)	Control Limit (L)	Chart Type	Slope Shift Factor (γ)	Intercept Shift Factor (α)	History Length In-Control	ARL_1
1.0	1.0	3.00	Shewhart	1.5	0σ	30	18.5505
0.2	1.0	2.86	EWMA	1.5	0σ	30	10.2605
0.2	5.0	3.0358	EWMA-*d*	1.5	0σ	30	9.3765
1.0	1.0	3.00	Shewhart	2.0	0σ	30	8.4340
0.2	1.0	2.86	EWMA	2.0	0σ	30	6.6010
0.2	5.0	3.0358	EWMA-*d*	2.0	0σ	30	6.3550
1.0	1.0	3.00	Shewhart	2.5	0σ	30	5.9900
0.2	1.0	2.86	EWMA	2.5	0σ	30	5.3390
0.2	5.0	3.0358	EWMA-*d*	2.5	0σ	30	5.1965
1.0	1.0	3.00	Shewhart	1.0	2σ	30	122.3605
0.2	1.0	2.86	EWMA	1.0	2σ	30	15.7090
0.2	5.0	3.0358	EWMA-*d*	1.0	2σ	30	9.2375
1.0	1.0	3.00	Shewhart	1.0	3σ	30	22.7975
0.2	1.0	2.86	EWMA	1.0	3σ	30	3.3280
0.2	5.0	3.0358	EWMA-*d*	1.0	3σ	30	3.1040

We conduct the same simulations for the case in which we only have 10 in-control observations to estimate our parameters from. We expect the out of control ARL to be larger in this case because the chart is not trained with as many in-control points. The results from the conditions of this simulation study can be seen in Table 4.

Table 4: Case 1 Out of Control ARL Simulation Results In-control History of 10 points

Weight (λ)	delay Parameter (d)	Control Limit (L)	Chart Type	Slope Shift Factor (γ)	Intercept Shift Factor (α)	History Length In-Control	ARL_1
1.0	1.0	3.00	Shewhart	1.5	0σ	10	299.7390
0.2	1.0	2.86	EWMA	1.5	0σ	10	102.9095
0.2	2.0	2.9339	EWMA-*d*	1.5	0σ	10	71.0905
1.0	1.0	3.00	Shewhart	2.0	0σ	10	93.1250
0.2	1.0	2.86	EWMA	2.0	0σ	10	10.1320
0.2	2.0	2.9339	EWMA-*d*	2.0	0σ	10	8.9280
1.0	1.0	3.00	Shewhart	2.5	0σ	10	25.6075
0.2	1.0	2.86	EWMA	2.5	0σ	10	6.8780
0.2	2.0	2.9339	EWMA-*d*	2.5	0σ	10	6.1780
1.0	1.0	3.00	Shewhart	1.0	2σ	10	238.4975
0.2	1.0	2.86	EWMA	1.0	2σ	10	41.4040
0.2	2.0	2.9339	EWMA-*d*	1.0	2σ	10	29.6540
1.0	1.0	3.00	Shewhart	1.0	3σ	10	110.4595
0.2	1.0	2.86	EWMA	1.0	3σ	10	12.7190
0.2	2.0	2.9339	EWMA-*d*	1.0	3σ	10	9.6665

After testing Case I we wanted to also conduct the same tests but this time with unknown in-control mean and unknown in-control standard deviation. This result will give us a better idea of how the chart will perform in practice because in most self-starting chart monitoring scenario the engineer would not know the value of the in-control parameters prior to the start of the run. Since it is more difficult to detect a shift when both the in-control parameters are unknown instead of just the in-control mean we expect our out of control ARL to be greater across all tests. After running the simulations we present our results in Tables 5 and 6.

Once again we run our first set of simulations with 30 in-control points and then also test the same cases with 10 in-control observations.

Table 5: Case 2 Out of Control ARL Simulation Results In-control History of 30 points

Weight (λ)	delay Parameter (d)	Control Limit (L)	Chart Type	Slope Shift Factor (γ)	Intercept Shift Factor (α)	History Length In-Control	ARL_1
1.0	1.0	3.00	Shewhart	1.5	0σ	30	>500
0.2	1.0	2.86	EWMA	1.5	0σ	30	10.9855
0.2	5.0	3.0358	EWMA-*d*	1.5	0σ	30	9.6440
1.0	1.0	3.00	Shewhart	2.0	0σ	30	>500
0.2	1.0	2.86	EWMA	2.0	0σ	30	6.8240
0.2	5.0	3.0358	EWMA-*d*	2.0	0σ	30	6.4525
1.0	1.0	3.00	Shewhart	2.5	0σ	30	>500
0.2	1.0	2.86	EWMA	2.5	0σ	30	5.5155
0.2	5.0	3.0358	EWMA-*d*	2.5	0σ	30	5.3545
1.0	1.0	3.00	Shewhart	1.0	2σ	30	>500
0.2	1.0	2.86	EWMA	1.0	2σ	30	28.0750
0.2	5.0	3.0358	EWMA-*d*	1.0	2σ	30	13.4465
1.0	1.0	3.00	Shewhart	1.0	3σ	30	>500
0.2	1.0	2.86	EWMA	1.0	3σ	30	6.6260
0.2	5.0	3.0358	EWMA-*d*	1.0	3σ	30	3.6455

We expect that the combined effect of having unknown in-control mean and standard deviation with the smaller number of in-control observations will increase our out of control ARLs even further. We show these results in Table 6.

Table 6: Case 2 Out of Control ARL Simulation Results In-control History of 10 points

Weight (λ)	delay Parameter (d)	Control Limit (L)	Chart Type	Slope Shift Factor (γ)	Intercept Shift Factor (α)	History Length In-Control	ARL_1
1.0	1.0	3.00	Shewhart	1.5	0σ	10	>500
0.2	1.0	2.86	EWMA	1.5	0σ	10	169.7330
0.2	2.0	2.9339	EWMA-*d*	1.5	0σ	10	121.7490
1.0	1.0	3.00	Shewhart	2.0	0σ	10	>500
0.2	1.0	2.86	EWMA	2.0	0σ	10	25.3455
0.2	2.0	2.9339	EWMA-*d*	2.0	0σ	10	15.2350
1.0	1.0	3.00	Shewhart	2.5	0σ	10	>500
0.2	1.0	2.86	EWMA	2.5	0σ	10	8.5175
0.2	2.0	2.9339	EWMA-*d*	2.5	0σ	10	7.0600
1.0	1.0	3.00	Shewhart	1.0	2σ	10	>500
0.2	1.0	2.86	EWMA	1.0	2σ	10	124.3725
0.2	2.0	2.9339	EWMA-*d*	1.0	2σ	10	96.0830
1.0	1.0	3.00	Shewhart	1.0	3σ	10	>500
0.2	1.0	2.86	EWMA	1.0	3σ	10	63.6840
0.2	2.0	2.9339	EWMA-*d*	1.0	3σ	10	31.7305

In Tables 5 and 6 we notice that there are instances in which the out of control ARL exceeds 500 data points. This result comes as a surprise since the in-control ARL for all of these charts is approximately 370 so we would expect the out of control ARL to be below this value. This result leads us to the conclusion that the Shewhart *Q* chart fails in detecting the shift we have simulated, and may not be effective at all in detecting these kinds of small magnitude linear shifts. We notice that in the case presented in Tables 3 and 4, when the in-control variance is known, we do not observe this ineffectiveness. This leads us to believe that the ineffectiveness in detection stems from the estimation of the in-control process variance. We hypothesize that the estimate of the MSE is contaminated by out of control points and as a result the MSE estimate at each time *t* is too large because of the wide range of response we observe.

After reviewing the results from our simulation study we were able to establish some clear patterns in performance of the charting methods. First, we notice that the Shewhart chart was significantly less effective in shift detection than both of the EWMA based methods for most cases. It has similar performance to the EWMA based methods only for very large shifts. This conforms to our expectation since EWMA charts are much more effective in small shift detection than are Shewhart charts. Therefore, if small shift detection is desired we recommend either of our EWMA based methods over our Shewhart chart. Another noticeable trend is a product of the number of in-control observations seen prior to the change point. In general when 30 in-control observations were seen our methods tended to perform better than in the case when only 10 in-control observations were seen. This too does not come as a surprise to us. With more in-control observations the chart can better estimate the in-control state and is essentially trained more before the parameter shifts are introduced and therefore has better performance. Another observation we notice also involves the length of the in-control state. We observe that in Case 2 when both in-control mean and in-control standard deviation are unknown that our charts are more sensitive to the number of in-control observations. Since the in-control standard deviation needs to be estimated in this case along with the mean, less in-control observations make it even more difficult to detect shifts in the linear trend.

Especially when looking at the ARL_1 performance in the cases in which we introduced small shifts it is clear that the EWMA methods outperform the Shewhart method we propose. Between EWMA and EWMA-d, they perform similarly with

EWMA-d always slightly better than EWMA. For more difficult cases when the shift is small and the ARL_1 is large, the advantage of EWMA-d becomes more noticeable.

3.5 Application to Battery Degradation Data

After testing our proposed methods on simulated data we wanted to test our methods on an actual collection of data that exemplifies the type of degradation trend that we wish to monitor. This example involves monitoring the changing internal resistance signal of automotive lead-acid batteries in continued week by week use. The internal resistance is considered a critical measure of the health of these batteries so this internal resistance (mΩ) measure will serve as our quality characteristic of interest and our estimation of the current level of degradation of the battery. In general an increasing internal resistance measure indicates decreasing battery health. This resistance measure often increases as the time the battery is in use increases which is why it serves as a useful indicator to predict the future health status of the battery.

In assessing the battery life based on the resistance value we are interested in detecting the point in which the degradation trend shifts from its in-control state into its out of control state. It has been observed that the internal resistance measure evolves differently towards the end of a battery's life. After a battery has degraded down to a certain level the resistance trend may change its form and evolve to a more accelerated degradation trend compared to its early stage of usage, which is what we call the change point. Once a battery is in this accelerated degradation phase of its lifetime, failure is expected soon after. It is therefore in our best interest to detect this change point as soon as possible so we can take precautionary measures and replace the battery in order to avoid system failure.

There are two different kinds of failures that we consider in failure analysis, soft failure and hard failure. Soft failure is defined as the event of the degradation signal surpassing a certain threshold. This soft failure threshold gives us an idea of when to expect failure but the actual failure of the battery differs from battery to battery. This is why we desire a more precise method for predicting the occurrence of failure, which we accomplish by detecting the shift in degradation trend. The time when the battery fails to crank the engine is what we consider to be the actual failure state of the battery, which is referred to as hard failure. From looking at internal resistance and the occurrence of hard failure it is made obvious that there is no specific fixed threshold that determines the failure time. Therefore, our methods that detect the occurrence of the change point into the accelerated degradation state of the battery will offer a warning to the engineer before hard failure actually occurs. This prognosis of battery health in the near future will save the engineer time and money since preventative action can help avoid system failure and its associated costs.

We take a closer look at internal resistance data collected weekly for two different batteries what we will call JBI1 and JBI7. In Figure 4(a) we show the JBI1 data plotted on a graph of Internal Resistance vs. Time. It can be observed that this data exhibits a linear trend that changes in value at some change point to a more accelerated trend. It is our desire to detect when this change point occurs so that we know when the process goes from the in-control to the out of control state. We show in Figure 4(b) our Shewhart Chart method, in Figure 4(c) our EWMA based method and in Figure 4(d) our EWMA with delay parameter d=2.

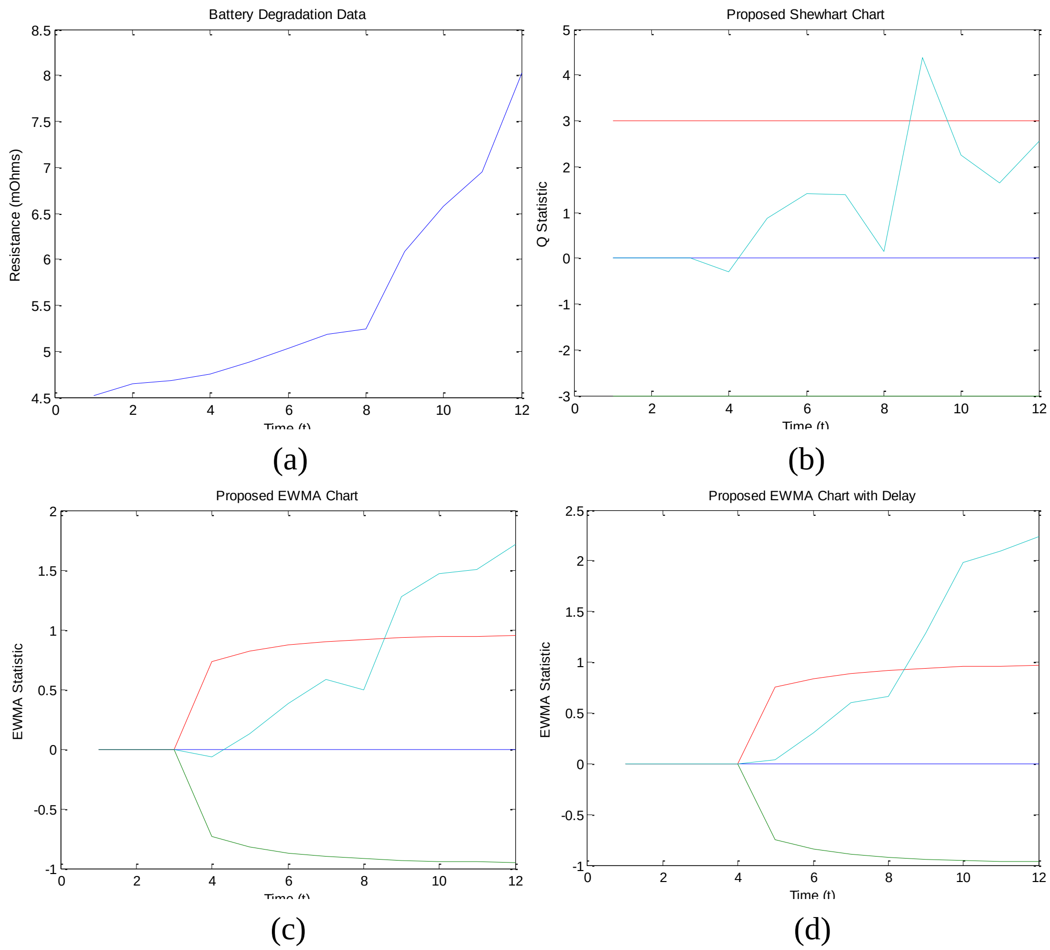


Figure 4: Application of our Charting Methods to Battery JBI1 Data

From the charts in Figure 4 one can observe when the chart indicated an out of control signal by noting the point in which the magnitude of the statistic exceeded the control limits. We call this point the signal point and have included our findings in Table 7 for each of our proposed charting methods.

Table 7: Signal Points of our Proposed Charts for JBI1

Weight ($\boldsymbol{\lambda}$)	delay Parameter (d)	Control Limit (L)	Chart Type	Signal Point
1.0	1.0	3.00	Shewhart	9
0.2	1.0	2.86	EWMA	9
0.2	2.0	2.9339	*EWMA-d*	9

We can see from the results we obtained that all three of the charts were able to detect the occurrence of the changing linear trend quickly and effectively. Since they all signaled at time t=9 this consistency shows that these would all be legitimate methods given the right conditions.

We then take a look at another battery degradation data set. In Figure 5(a) we show the JBI7 data plotted on a graph of Internal Resistance vs. Time. It can be observed that this data also exhibits a linear trend that changes in value at some change point to a more accelerated trend. Like before we show in Figure 5(b) our Shewhart Chart method, in Figure 5(c) our EWMA based method and in Figure 5(d) our EWMA with delay parameter d=2.

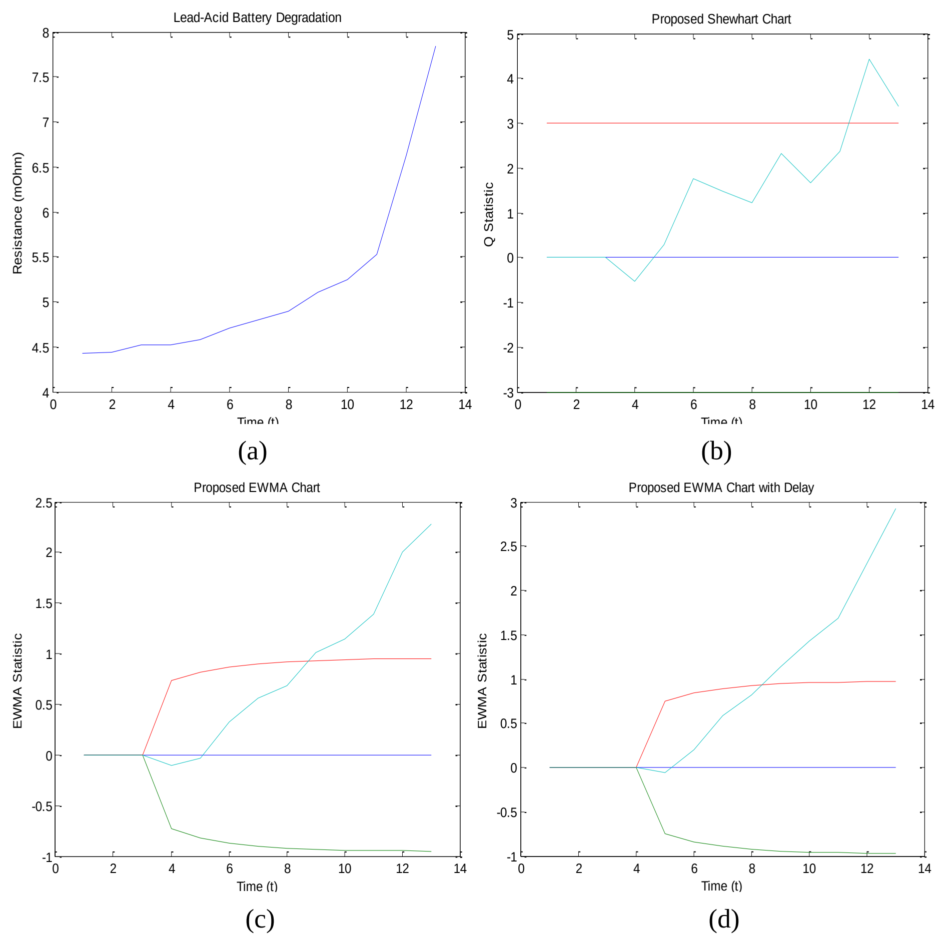


Figure 5: Application of our Charting Methods to Battery JBI7 Data

From the charts in Figure 5 we once again record the signal point for each of the charts and have included our findings in Table 8 for each of our proposed charting methods.

Table 8: Signal Points of our Proposed Charts for JBI7

Weight (λ)	delay Parameter (d)	Control Limit (L)	Chart Type	Signal Point
1.0	1.0	3.00	Shewhart	12
0.2	1.0	2.86	EWMA	9
0.2	2.0	2.9339	*EWMA-d*	9

We can see from the results we obtained that all three of the charts were able to detect the occurrence of the new trend but this time the performance differed among methods. The Shewhart chart did not signal until the time t=12 while both the *EWMA* based charts both signaled at time t=9. Although the biggest change in rate occurs at about t=11, by looking at the degradation signal in Figure 5 more closely, accelerated degradation seems to occur at about t=9. But the Shewhart chart fails to detect this acceleration since the change of the degradation rate is small. This may cause a missed opportunity to plan preventive action before the actual failure occurs. This shows that in the case of shifts that are smaller in magnitude, harder to detect, that there is a clear advantage in using one of the *EWMA* methods we proposed instead of the Shewhart based method.

Since there are a limited amount of data points available in these battery examples this is a typical situation where a self-starting chart is especially helpful. With such limited data availability one cannot establish in-control estimates due to data collection limitations, therefore the in-control parameters must be established on-line while the process is happening. We confirm that our online estimation of the parameters is valid through our results from these battery examples and through simulation study.

CHAPTER 4 CONCLUSION AND FUTURE WORK

In this thesis we proposed three methods to monitor a non-stable in-control mean that follows some linear trend. Our three methods included a Shewhart Chart method, a EWMA based method and a EWMA with delay parameter *d*. All of our methods utilized the ideas of the self-starting *Q* chart. After reviewing the current SPC literature we found there was an opportunity to solve the problem of monitoring situations in which the in-control state follows a linear trend so there are very little research efforts to solve this problem.

We start by reviewing the self-starting *Q* chart along with its assumptions and derivation. We let the development of Quesenberry's *Q*-statistic guide us in developing our own *Q* statistic that utilized ideas from linear regression in order to monitor a non-stable mean. After developing our own *Q* statistic we were able to make adjustments that would improve upon the most basic implementation of the *Q* statistic for linear trend. We also propose two methods that utilize a *EWMA* scheme. We expect these charts to be less subject to the contamination effects on our in-control estimation from our out of control state and also expect them to be more effective in small shift detection. We verified both of these results through simulation study and in applying our methods to battery degradation data. In our third method we introduce a delay parameter *d* to our statistic in addition to applying *EWMA.* We introduce this delay parameter to further reduce the contamination effects we mentioned earlier and verified that this method performed the best out of our methods for small shift detection. Our Shewhart chart method is still a valid option for solving this problem but is more effective in cases where the shift experienced is larger in magnitude. We recommend the use of our EWMA methods for the small shift detection case as they are more sensitive. Note that although in this thesis the explanatory variable is time *t*, all the results can be easily extended when a general explanatory variable is used.

After gathering results from our research we realized several opportunities for future work. One such opportunity is extending the simple linear regression case we presented to one in which there are two or more explanatory variables. In practice it is common for a process to have multiple explanatory variables that can affect the quality measurements. Therefore, for the practicality of these charting methods it would be beneficial for the chart to be able to handle the multiple linear regression case so we can monitor linear trends that depend on two or more variables.

Another obvious extension of our method is to monitor other in-control trends. There are several processes in which the in-control state follows a trend that is non-linear. Being able to monitor a different trend would have all the same advantages of those for the linear trend case and would add significant flexibility to our proposed methods.

Like what is done in many *EWMA* papers, there is an additional opportunity for future work in establishing in-control ARL values for the varying levels of the smoothing parameter since in this thesis we only considered the case in which λ=0.2. This would help greatly when an engineer wants to use such a chart in practice since it can be very time consuming to estimate ARL values through Monte Carlo simulation.

We also noticed opportunities that stem from relaxing some of our assumptions. For our charting methods we assumed that the observations were i.i.d. coming from a normal distribution. There are cases in which the data have autocorrelation and a charting method that could handle this situation would be very beneficial. The other assumption made is that the data is normally distributed, this too presents an opportunity for further work to handle non-normal data.

APPENDIX

We provide in this appendix the derivation of the distribution-preserving constant used in Equations (3-7) and (3-10). We substitute $\hat{\beta}_1^{(t-d)} = \frac{\sum_{i=1}^{t-d}(X_i-\bar{X}_{t-d})y_i}{\sum_{i=1}^{t-d}(X_i-\bar{X}_{t-d})^2}$ into the equation and factor the summation out front in order to simplify our calculation of the variance.

$$V\big(Y_t - \hat{Y}_{t-d}(d)\big) = \sigma^2 + V\left(\sum_{i=1}^{t-d}\{\frac{y_i}{t-d} + (X_t - \bar{X}_{t-d})\frac{(X_i - \bar{X}_{t-d})y_i}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\}\right)$$

$$= \sigma^2 + V\left(\sum_{i=1}^{t-d}\{\frac{1}{t-d} + \frac{(X_i - \bar{X}_{t-d})(X_t - \bar{X}_{t-d})}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\}y_i\right)$$

We then define

$$C_i = \left(\frac{1}{t-d} + \frac{(X_i - \bar{X}_{t-d})(X_t - \bar{X}_{t-d})}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right)$$

$$V\big(Y_t - \hat{Y}_{t-d}(d)\big) = \sigma^2 + V(\textstyle\sum_{i=1}^{t-d} C_i\, y_i)$$

$$V\big(Y_t - \hat{Y}_{t-d}(d)\big) = \sigma^2 + \sum_{i=1}^{t-d} {C_i}^2\, \sigma^2)$$

$$\sum_{i=1}^{t-d} {C_i}^2 = \sum_{i=1}^{t-d}\left(\frac{1}{t-d} + \frac{(X_i - \bar{X}_{t-d})(X_t - \bar{X}_{t-d})}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right)^2$$

$$= \sum_{i=1}^{t-d}\left(\frac{1}{t-d}\right)^2 + \frac{2(X_t - \bar{X}_{t-d})\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})}{(t-d)\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2} + \frac{(X_t - \bar{X}_{t-d})^2\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}{(\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2)^2}$$

$\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d}) = 0$ so the second term above drops out

$$\sum_{i=1}^{t-1} {C_i}^2 = \frac{1}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-1}(X_i - \bar{X}_{t-d})^2}$$

$$V\big(Y_t - \hat{Y}_{t-d}(d)\big) = \sigma^2 + \sum_{i=1}^{t-1} {C_i}^2\, \sigma^2$$

$$= \sigma^2 + \sigma^2\left[\frac{1}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d}(X_i - \bar{X}_{t-d})^2}\right]$$

$$= \sigma^2 \left[1 + \left[\frac{1}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d} (X_i - \bar{X}_{t-d})^2} \right] \right]$$

$$= \sigma^2 \left[\frac{t-d}{t-d} + \left[\frac{1}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d} (X_i - \bar{X}_{t-d})^2} \right] \right]$$

$$= \sigma^2 \left[\left[\frac{t}{t-d} + \frac{(X_t - \bar{X}_{t-d})^2}{\sum_{i=1}^{t-d} (X_i - \bar{X}_{t-d})^2} \right] \right]$$

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